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# Hypergeometric States and Their Nonclassical Properties

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## Abstract

‘Hypergeometric states’, which are a one-parameter generalization of binomial states of the single-mode quantized radiation field, are introduced and their nonclassical properties are investigated. Their limits to the binomial states and to the coherent and number states are studied. The ladder operator formulation of the hypergeometric states is found and the algebra involved turns out to be a one-parameter deformation of  $su(2)$  algebra. These states exhibit highly nonclassical properties, like sub-Poissonian character, antibunching and squeezing effects. The quasiprobability distributions in phase space, namely the  $Q$  and the Wigner functions are studied in detail. These remarkable properties seem to suggest that the hypergeometric states deserve further attention from theoretical and applicational sides of quantum optics.

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# 1 Introduction

The number and the coherent states of the quantized radiation field play important roles in quantum optics and are extensively studied [1]. The binomial states (BS, also called intermediate number-coherent states) introduced by Stoler, Saleh and Teich in 1985 [2], interpolate between the *most nonclassical* number states and the *most classical* coherent states, and reduce to them in two different limits. Some of their properties [2, 3, 4], methods of generation [2, 3, 5], as well as their interaction with atoms [6], have been investigated in the literature. The BS is defined as a linear superposition of number states ( $|n\rangle$ ,  $n = 0, 1, \dots$ ) in an  $M + 1$ -dimensional subspace

$$|\eta, M\rangle = \sum_{n=0}^M \beta_n^M(\eta) |n\rangle, \quad (1.1)$$

where  $\eta$  is a real parameter satisfying  $0 < \eta < 1$  (probability), and

$$\beta_n^M(\eta) = \left[ \binom{M}{n} \eta^n (1-\eta)^{M-n} \right]^{1/2}. \quad (1.2)$$

The name ‘binomial state’ comes from the fact that their photon distribution  $|\langle n|\eta, M\rangle|^2 = |\beta_n^M(\eta)|^2$  is simply the binomial distribution with probability  $\eta$ . In the two limits  $\eta \rightarrow 1$  and  $\eta \rightarrow 0$  it reduces to number states:

$$|1, M\rangle = |M\rangle, \quad |0, M\rangle = |0\rangle. \quad (1.3)$$

In a different limit of  $M \rightarrow \infty$ ,  $\eta \rightarrow 0$  with  $\eta M = \alpha^2$  fixed ( $\alpha$  real constant),  $|\eta, M\rangle$  reduces to the coherent states (not the most general ones, only those with real amplitude  $\alpha$ ), which corresponds to the Poisson distribution in probability theory[7]. It is well known that the binomial distribution tends to the Poisson distribution in the above limit[7]. The notion of BS was also generalized to the intermediate squeezed states [8] and the number-phase states [9], as well as their  $q$ -deformation [10]. In particular, in a previous paper [11] we derived the ladder-operator form the BS and on this basis we generalized the BS to the generalized BS which possessed the number and squeezed states as limits in the framework of Lie algebra  $su(2)$ .

In the present paper we shall propose a one-parameter generalization of the binomial states, the *hypergeometric states* (HGS), whose photon distribution is the hypergeometric distribution in probability theory [7]. It is well known that the hypergeometric distribution tends to the binomial distribution in certain limit. This leads to the reduction of HGS to the BS in the same limit. Some mathematical properties, such as the

equivalent ladder operator definition, related algebraic structures, will be formulated. It is interesting that the algebraic structure is a well-investigated *generally deformed oscillator algebra* [12], which reduces to the universal enveloping algebra of Lie algebra  $su(2)$ , the algebraic structure characterizing the binomial states. The nonclassical properties of the HGS, the photon statistical properties, sub-Poissonian distribution, antibunching effect and the squeezing effect will be investigated in detail. Two well known quasi-probability distributions, the  $Q$ -function and Wigner function will be evaluated to study the nonclassical properties. It will be shown that the HGS exhibit highly nonclassical behaviour.

## 2 HGS and basic properties

### 2.1 Definition

The HGS is defined as a linear combination of number states in an  $(M + 1)$ -dimensional subspace

$$|L, M, \eta\rangle = \sum_{n=0}^M H_n^M(\eta, L)|n\rangle, \quad (2.1)$$

where the *probability*  $\eta$  is a real parameter satisfying  $0 < \eta < 1$ .  $L$  is a real number satisfying

$$L \geq \max\{M\eta^{-1}, M(1 - \eta)^{-1}\}, \quad (2.2)$$

and

$$H_n^M(\eta, L) = \left[ \binom{L\eta}{n} \binom{L(1 - \eta)}{M - n} \right]^{\frac{1}{2}} \left( \frac{L}{M} \right)^{-\frac{1}{2}}, \quad (2.3)$$

$$\binom{\alpha}{n} = \frac{\alpha(\alpha - 1) \cdots (\alpha - n + 1)}{n!}, \quad \binom{\alpha}{0} \equiv 1. \quad (2.4)$$

Note that in Eq.(2.4) the real number  $\alpha$  is not necessarily an integer.

The name of HGS comes from the fact that the photon distribution  $|\langle n | L, M, \eta \rangle|^2$

$$|\langle n | L, M, \eta \rangle|^2 = [H_n^M(\eta, L)]^2, \quad (2.5)$$

is the hypergeometric distribution in probability theory [7]. (For a background, see the Appendix.) We remark that in the case of  $M = 1$ , the HGS  $|L, 1, \eta\rangle$  is  $L$  independent and is equal to the binomial state  $|L, 1, \eta\rangle \equiv |1, \eta\rangle$ .

It is well known that the hypergeometric distribution tends to the binomial distribution in the limit  $L \rightarrow \infty$ . So, correspondingly, the HGS tends to the BS in this limit

$$|L, M, \eta\rangle \xrightarrow{L \rightarrow \infty} |M, \eta\rangle. \quad (2.6)$$

This fact can be verified directly. Furthermore, the BS go to the number states and coherent states, the latter corresponds to the Poisson distribution, in certain limits. So the HGS reduce to the number and coherent states in these limits:

$$|L, M, \eta\rangle \xrightarrow{L \rightarrow \infty} |M, \eta\rangle \longrightarrow \begin{cases} |M\rangle, & \text{when } \eta \rightarrow 1, \\ |0\rangle, & \text{when } \eta \rightarrow 0, \\ |\alpha\rangle, & \text{when } M \rightarrow \infty, \eta \rightarrow 0 \text{ with finite } \eta M \equiv \alpha. \end{cases} \quad (2.7)$$

It is easy to show that the HGS are normalized by using the following identity

$$\sum_{n=0}^M \binom{\alpha}{n} \binom{\beta}{M-n} = \binom{\alpha + \beta}{M}, \quad (2.8)$$

which can be obtained by comparing the power series expansion of  $(1+t)^\alpha(1+t)^\beta = (1+t)^{\alpha+\beta}$ .

## 2.2 Ladder operator form

In a previous paper [11] we have shown that the BS admit the ladder operator formulation, namely, they are characterized by the following eigenvalue equation

$$[\sqrt{\eta}N + \sqrt{1-\eta}J_M^+]|M, \eta\rangle = \sqrt{\eta}M|M, \eta\rangle, \quad (2.9)$$

where  $J_M^+ \equiv \sqrt{M-N}a$  is the raising operator of the Lie algebra  $su(2)$  via its Holstein-Primakoff realization. Hereafter  $a^\dagger$  and  $a$  are the creation and annihilation operators of the photon and  $N = a^\dagger a$  is the number operator. So we naturally expect that the HGS satisfy a generalized eigenvalue equation and the algebra involved is a deformation of  $su(2)$ . To this end, we suppose that the HGS satisfy an eigenvalue equation

$$[f(N) + g(N)a]|L, M, \eta\rangle = \lambda|L, M, \eta\rangle, \quad (2.10)$$

in which  $\lambda$  is the eigenvalue to be determined. Inserting (2.1) into (2.10) and comparing the coefficients, we obtain

$$f(M) = \lambda, \quad (2.11)$$

$$\begin{aligned} (\lambda - f(n)) \left[ \binom{L\eta}{n} \binom{L(1-\eta)}{M-n} \right]^{\frac{1}{2}} = \\ \left[ \binom{L\eta}{n+1} \binom{L(1-\eta)}{M-n-1} (n+1) \right]^{\frac{1}{2}} g(n), \quad (0 \leq n \leq M-1). \end{aligned} \quad (2.12)$$

From (2.11) and (2.12), we have

$$f(M) - f(n) = \sqrt{\frac{L\eta - n}{L(1-\eta) - M + n + 1}} \sqrt{M-n} g(n). \quad (2.13)$$

Observe that in the right side of the above equation  $M$  and  $n$  appear in the form  $M - n$ . Requiring that Eq.(2.10) reduces to (2.9) in the limit  $L \rightarrow \infty$ , we obtain

$$f(N) = \sqrt{\eta}N, \quad g(N) = \sqrt{\eta} \left[ \frac{L(1-\eta) - M + N + 1}{L\eta - N} \right]^{\frac{1}{2}} \sqrt{M - N}. \quad (2.14)$$

Substituting (2.14) into (2.10) we arrive at the ladder operator form of HGS

$$\left[ \sqrt{\eta}N + \sqrt{\eta} \left( \frac{L(1-\eta) - M + N + 1}{L\eta - N} \right)^{\frac{1}{2}} J_M^+ \right] |L, M, \eta\rangle = \sqrt{\eta}M |L, M, \eta\rangle. \quad (2.15)$$

It is easy to see that (2.15) reduces to (2.9) in the limit of  $L \rightarrow \infty$  for finite  $M$  and  $N$ .

Here we would like to remark that the operator in the left side of (2.15) is an  $(M + 1) \times (M + 1)$  matrix and it generally has  $M + 1$  eigenvalues and eigenstates. The HGS is only one eigenstate of the eigenvalue  $\sqrt{\eta}M$ .

Let us examine the algebraic structure involved in (2.15). Define  $\mathcal{A}(L, M)$  as an associative algebra with generators

$$N, \quad A_M^- = \left( \frac{\eta}{1-\eta} \right)^{\frac{1}{2}} \left( \frac{L(1-\eta) - M + N + 1}{L\eta - N} \right)^{\frac{1}{2}} J_M^+, \quad A_M^+ = (A_M^-)^\dagger. \quad (2.16)$$

Then it is easy to verify that these operators satisfy the following commutation relations

$$[N, A_M^\pm] = \pm A_M^\pm, \quad A_M^+ A_M^- = F(N), \quad A_M^- A_M^+ = F(N + 1), \quad (2.17)$$

where the function  $F(N)$

$$F(N) = \frac{\eta[L(1-\eta) - M + N]N(M - N + 1)}{(1-\eta)(L\eta - N + 2)}, \quad (2.18)$$

is non-negative for  $0 \leq N \leq M$ . This algebra  $\mathcal{A}(L, M)$  is nothing but the *generally deformed oscillator algebra* with the structure function  $F(N)$  [12]. In terms of the generators of  $\mathcal{A}(L, M)$ , Eq.(2.15) can be rewritten as

$$\left[ \sqrt{\eta}N + \sqrt{1-\eta}A_M^- \right] |L, M, \eta\rangle = \sqrt{\eta}M |L, M, \eta\rangle. \quad (2.19)$$

It is interesting that this algebra  $\mathcal{A}(L, M)$  is an  $L$ -deformation of  $su(2)$  in the sense that it contracts to the universal enveloping algebra of the Lie algebra  $su(2)$  in the limit  $L \rightarrow \infty$  with  $M$  and  $N$  finite

$$A_M^\pm \xrightarrow{L \rightarrow \infty} J_M^\mp. \quad (2.20)$$

This means that the ladder operator form reduces to that of the BS.

### 3 Nonclassical Properties

In this section we turn to the nonclassical properties of the HGS.

#### 3.1 Mean photon number and fluctuation

The mean photon number in the HGS is obtained as

$$\langle N \rangle = \langle L, M, \eta | N | L, M, \eta \rangle = M\eta. \quad (3.1)$$

It is interesting that it is independent of  $L$  and therefore it is exactly same as that of the BS. However, the mean value of  $N^2$  depends on  $L$

$$\langle N^2 \rangle = M\eta \frac{L + L\eta M - L\eta - M}{L - 1}. \quad (3.2)$$

Then the fluctuation of the photon number is

$$\langle \Delta N^2 \rangle \equiv \langle N^2 \rangle - \langle N \rangle^2 = \eta(1 - \eta)M \frac{L - M}{L - 1} = \langle \Delta N^2 \rangle_{BS} \frac{L - M}{L - 1}, \quad (3.3)$$

where the  $\langle \Delta N^2 \rangle_{BS} \equiv \eta(1 - \eta)M$  is the corresponding fluctuation of the BS. We find that this fluctuation is always weaker than that of the BS since the factor  $W(L, M) = (L - M)/(L - 1)$  is always smaller than 1 except for  $M = 1$  and the limit  $L \rightarrow \infty$ . Let us go into some detail of the factor  $W(L, M)$ , which is referred to as the *weakening factor*. We shall see that this factor lies between one half and 1

$$\frac{1}{2} < W(L, M) < 1. \quad (3.4)$$

In fact, as a function of  $L$  for fixed  $M$ ,  $W(L, M)$  is an increasing function and the smallest  $W(L, M)$  corresponds to the smallest  $L$ , which is  $2M$  for  $\eta = 0.5$ . In this case the weakening factor is rewritten as  $W(M) = M/(2M - 1)$  which is always larger than 1/2.

So, in comparison with the BS, the fluctuation is reduced in the HGS. For large  $M$  with  $\eta = 0.5$ , the fluctuation is only about half as much as the binomial states. This is an important feature of the HGS.

#### 3.2 Sub-Poissonian distribution

Let us introduce Mandel's  $Q$  parameter [13] defined by

$$Q = \frac{\langle \Delta N^2 \rangle - \langle N \rangle}{\langle N \rangle}, \quad (3.5)$$

which measures the deviation from the Poisson distribution (the coherent state,  $Q = 0$ ). If  $Q < 0$  ( $> 0$ ), the field is called sub(super)-Poissonian, respectively. For the HGS, it is easy to find that

$$Q = (1 - \eta)W(L, M) - 1, \quad (3.6)$$

which is generally negative since  $1 - \eta < 1$  and  $W(L, M) < 1$ , namely, the field on the HGS is sub-Poissonian. Exceptions are the coherent state limit ( $L \rightarrow \infty$ ,  $M \rightarrow \infty$ ,  $\eta \rightarrow 0$  with  $\eta M = \alpha^2$  finite) and the vacuum state limit ( $L \rightarrow \infty$ ,  $\eta \rightarrow 0$ ). The extreme case is  $Q = -1$  since  $(1 - \eta)W(L, M)$  is always positive. In the case  $\eta \rightarrow 1$  and  $L = (1 - \eta)^{-1}M \rightarrow \infty$ , namely, on the number states, the extreme case occurs.

### 3.3 Antibunching effect

We say the field is antibunched if the second-order correlation function  $g^{(2)}(0)$  satisfies [14]

$$g^{(2)}(0) = \frac{\langle a^\dagger a^\dagger a a \rangle}{\langle a^\dagger a \rangle^2} = \frac{\langle N^2 - N \rangle}{\langle N \rangle^2} < 1. \quad (3.7)$$

For the HGS

$$g^{(2)}(0) = \frac{M - 1}{M} \frac{L\eta - 1}{L\eta - \eta} \quad (3.8)$$

which always satisfies the condition (3.7). So the HGS is antibunched. In fact, the occurrence of antibunching effect and sub-Poissonian are concomitant for single mode and time independent fields.

In the binomial state limit  $L \rightarrow \infty$ ,  $g^{(2)}(0)$  reduces to the second-order correlation function  $g^{(2)}(0)_{BS}$  of the BS

$$g^{(2)}(0) \xrightarrow{L \rightarrow \infty} g^{(2)}(0)_{BS} = \frac{M - 1}{M}. \quad (3.9)$$

Since the factor  $(L\eta - 1)/(L\eta - \eta) < 1$ , the HGS are more strongly antibunched than the BS.

### 3.4 Squeezing effect [1,4]

Define the quadrature operators  $x$  (coordinate) and  $p$  (momentum) by

$$x = \frac{1}{\sqrt{2}}(a^\dagger + a), \quad p = \frac{i}{\sqrt{2}}(a^\dagger - a). \quad (3.10)$$

Then their variances

$$\langle \Delta x^2 \rangle = \langle x^2 \rangle - \langle x \rangle^2, \quad \langle \Delta p^2 \rangle = \langle p^2 \rangle - \langle p \rangle^2, \quad (3.11)$$

obey the Heisenberg's uncertainty relation

$$\langle \Delta x^2 \rangle \langle \Delta p^2 \rangle \geq \frac{1}{4}. \quad (3.12)$$

If one of the  $\langle \Delta x^2 \rangle$  and  $\langle \Delta p^2 \rangle$  is smaller than  $1/2$ , the squeezing occurs. For convenience, we define the squeezing indices

$$S_x = \frac{\langle \Delta x^2 \rangle - 1/2}{1/2}, \quad S_p = \frac{\langle \Delta p^2 \rangle - 1/2}{1/2}. \quad (3.13)$$

If  $S_x < 0$  ( $S_p < 0$ ), there is squeezing in the quadrature  $x$  ( $p$ ). Now let us evaluate these indices.

It is easy to derive that

$$\begin{aligned} a^n |L, M, \eta\rangle &= \left(\frac{L}{M}\right)^{-1/2} \sqrt{L\eta(L\eta-1)\cdots(L\eta-n+1)} \\ &\times \sum_{k=0}^{M-n} \left[ \left(\frac{L\eta-n}{k}\right) \left(\frac{L(1-\eta)}{M-n-k}\right) \right]^{1/2} |k\rangle \end{aligned} \quad (3.14)$$

for  $n \leq M$  and  $a^n |L, M, \eta\rangle = 0$  for  $n > M$ . In particular, for  $n = 1, 2$ , we write

$$a |L, M, \eta\rangle = \sum_{k=0}^{M-1} \bar{H}_k |k\rangle, \quad a^2 |L, M, \eta\rangle = \sum_{k=0}^{M-2} \tilde{H}_k |k\rangle, \quad (3.15)$$

where

$$\begin{aligned} \bar{H}_k &= \sqrt{L\eta} \left(\frac{L}{M}\right)^{-\frac{1}{2}} \left[ \left(\frac{L\eta-1}{n}\right) \left(\frac{L(1-\eta)}{M-1-k}\right) \right]^{\frac{1}{2}}, \quad (0 \leq k \leq M-1), \\ \tilde{H}_k &= \sqrt{L\eta(L\eta-1)} \left(\frac{L}{M}\right)^{-\frac{1}{2}} \left[ \left(\frac{L\eta-2}{n}\right) \left(\frac{L(1-\eta)}{M-2-k}\right) \right]^{\frac{1}{2}}, \\ &\quad (0 \leq k \leq M-2). \end{aligned} \quad (3.16)$$

In terms of  $\bar{H}_n$  and  $\tilde{H}_n$ , we can obtain the squeezing indices as

$$S_x = 2 \sum_{n=0}^{M-2} H_n \tilde{H}_n + 2M\eta - 4 \left[ \sum_{n=0}^{M-1} H_n \bar{H}_n \right]^2, \quad (3.17)$$

$$S_p = 2M\eta - \sum_{n=0}^{M-2} H_n \tilde{H}_n, \quad (3.18)$$

in which  $H_n \equiv H_n^M(\eta, L)$  in (2.3). We have also suppressed the  $\eta$  and  $L$  dependence in  $\bar{H}_n$  and  $\tilde{H}_n$ .

Figures 1 and 2 are plots showing how the  $S_x$  depends on the parameter  $L$  and  $\eta$ . In each case, different values of  $M$  (5 and 50) are chosen. From these plots we find that:

(1). Dependence on  $L$  (Fig.1): When  $\eta$  or  $1 - \eta$  is small,  $|S_x|$  is always larger than that of the BS. The smaller  $L$ , the larger  $|S_x|$ . However, the difference from the BS is also small. This is understandable since in this case,  $L$  must be much larger than  $M$  due to the condition (2.2), and therefore the HGS are close to the BS. When  $\eta$  is around 0.5, the  $L$  can be closest to  $M$  (two times), and the HGS are far different from the BS. In particular, when  $L$  is small (close to  $2M$ ),  $S_x$  changes drastically in comparison with those of the BS. In general the squeezing is great for large  $M$  (50, see Fig.1(b)) and it decreases for small  $M$  (5, see Fig.1(a)).

(2). Dependence on  $\eta$  (Fig.2): We have chosen  $\eta = 0.25 \sim 0.75$  for  $M = 5$  and  $\eta = 0.05 \sim 0.95$  for  $M = 50$ . Similar to the BS, the squeezing increases as  $\eta$  increases to a maximal point, and then it decreases rapidly. This similarity is easy to understand. In order to have a wide range of  $\eta$ ,  $L$  is much larger than  $M$ . In this case, the HGS are closer to the BS. We find that the HGS exhibits stronger squeezing than the BS.

(3). Dependence on  $M$ : From Fig.1 we find that when  $\eta$  is around 0.5, and for small  $L$ , the squeezing is weaken for small  $M$  (5) than large  $M$  (50). From Fig.2 we conclude that large  $M$  has wider and stronger squeezing range of  $\eta$  than the small  $M$ . The HGS has wider squeezing range of  $\eta$  than the BS. This can be seen from Fig.1 with  $\eta = 0.72$  (a) and  $\eta = 0.923$  (b).

## 4 $Q$ and Wigner functions

The quasi-probability distributions [15] in the coherent state basis turn out to be useful measures for studying the nonclassical features of the radiation field. In this section we shall study the  $Q$  and Wigner functions.

### 4.1 $Q$ -function

We start with the  $Q(\beta)$  function

$$Q(\beta) = \frac{1}{\pi} |\langle \beta | L, M, \eta \rangle|^2, \quad (4.1)$$

where  $|\beta\rangle$  is the coherent states of the field. Substituting the HGS into (4.1) we obtain the  $Q$ -function as follows

$$Q(\beta) = \frac{\exp(-|\beta|^2)}{\pi} \left| \sum_{n=0}^M H_n^M(\eta, L) \frac{\beta^n}{\sqrt{n!}} \right|^2. \quad (4.2)$$

Here  $\beta$  is a complex  $c$ -number  $\beta = x + iy$ , with  $(x, y \equiv p)$  corresponding to the two quadrature operators  $x$  and  $p$  in (3.10).

Now we would like to investigate numerically the changes of  $Q$ -function for different  $L$ ,  $M$  and  $\eta$ . Fig.3 are plots of  $Q$ -function of HGS for different  $L$ , for fixed  $M = 5$ ,  $\eta = 0.5$ . When  $L \rightarrow \infty$ , the HGS is in fact a binomial state and its  $Q$ -function is shown in Fig.3(d) (see also [4]). Then we choose finite  $L$  values 40 (Fig.3(c)), 20 (Fig.3(b)) and 10 (Fig.3(a)) (note that 10 is the smallest allowed value of  $L$  for  $M = 5$ ,  $\eta = 0.5$ ). We can see clear deformation of the  $Q$ -function. At first sight this deformation pattern appears similar to that of  $Q$  function with respect to  $\eta$  given in Fig.4 of [4]. However, they are essentially different: Increase in  $\eta$  brings the gain of the energy or the mean photon number as given (3.1), while the changes of  $L$  does not correspond to any change of the mean energy due to Eq.(3.1). Fig.3(e) is the  $Q$ -function of HGS for  $\eta = 0.9$ ,  $L = 50$  and  $M = 5$ . This  $Q$ -function is almost the same as that of BS (see Fig.4(c) in Ref.[4]), as expected.

From the  $Q$ -functions we can also study the squeezing properties by examining the deformation of their contours. As before we pay our attention to the case  $\eta = 0.5$  and explain the drastic changes of squeezing for small  $L$ . Fig.4 are the plots of contours of  $Q$ -function for  $M = 5$ ,  $\eta = 0.5$  and  $L = 10, 28$  and  $\infty$  (binomial states). We find that, when we decrease  $L$ , the contour is first squeezed (Fig.4(b)) in the  $x$  direction until a maximum squeezing is reached. Then the contour deforms to the shape of an *ear*, which occupies a wider range in the  $x$  direction and the squeezing is reduced (Fig.4(a)). From this approach we can also explain the drastic increase of squeezing for  $M = 50$  and  $\eta = 0.5$ . In fact, when  $L$  becomes smaller, the shape of the contour is compressed, which corresponds to the strong squeezing. And this change continues until the smallest value of  $L$  allowed for  $\eta = 0.5$ , ie.  $L = 100$ . However, in contrast to the case  $M = 5$ , the case  $M = 50$  does not give rise to the shape of an *ear* (Fig.4(d)).

## 4.2 Wigner function

The Wigner function in the series form is defined as [16]

$$W(\beta) = \frac{2}{\pi} \sum_{k=0}^{\infty} (-1)^k \langle \beta, k | \rho | \beta, k \rangle, \quad (4.3)$$

where  $|\beta, k\rangle = D(\beta)|k\rangle \equiv \exp(\beta a^\dagger - \beta^* a)|k\rangle$ ,  $\rho$  is the density matrix (projector on the HGS) and takes the form

$$\rho = |L, M, \eta\rangle \langle L, M, \eta| \quad (4.4)$$

for the case in hand. It is easy to compute that

$$W(\beta) = \frac{2}{\pi} \sum_{k=0}^{\infty} (-1)^k \left| \sum_{n=0}^M H_n^M(\eta, L) \chi_{nk}(\beta) \right|^2. \quad (4.5)$$

Here  $\chi_{nk}(\beta) = \langle n | D(\beta) | k \rangle$  are given by [17]

$$\chi_{nk}(\beta) = \begin{cases} \sqrt{\frac{k!}{n!}} \exp(-|\beta|^2/2) \beta^{n-k} \mathcal{L}_k^{n-k}(|\beta|^2) & \text{if } n \geq k, \\ \sqrt{\frac{n!}{k!}} \exp(-|\beta|^2/2) (\beta^*)^{k-n} \mathcal{L}_n^{k-n}(|\beta|^2) & \text{if } n \leq k, \end{cases} \quad (4.6)$$

where  $\mathcal{L}_n^\alpha(|\beta|^2)$  are the generalized Laguerre polynomials.

Since the case  $M = 1$  is simply the binomial state and its Wigner functions have been investigated in detail in Ref.[4], we here consider the simplest nontrivial case:  $M = 2$ . Fig.5 are some plots of the Wigner function of HGS for  $M = 2$  and different  $\eta$ : (a)  $\eta = 0.2$  (b)  $\eta = 0.5$ , (c)  $\eta = 0.9$  and (d) the number state  $|2\rangle$  ( $\eta \rightarrow 1$  and  $L \rightarrow \infty$ ). In each case we choose the smallest possible value of  $L$  for given  $\eta$  and  $M$  to see the maximal contrast with the BS. We note that the case  $\eta \rightarrow 0$  is just the vacuum state and its Wigner function is simply the Gaussian centered at the origin. As  $\eta$  increases from 0, this Gaussian distribution continuously deforms to the Wigner function of  $|2\rangle$ , as shown in Fig.5. From  $\eta = 0.2$ , the negative parts of the Wigner functions are very clearly visible and this signifies the nonclassical properties.

Fig.6 are two plots of the Wigner function of the BS for  $M = 2$  and (b')  $\eta = 0.5$  and (c')  $\eta = 0.9$ . Comparing them with those of HGS, namely, Fig.5 (b,c), we find that in the case  $\eta = 0.5$  the Wigner distributions of HGS and BS are markedly different: distribution of HGS has two negative peaks while the BS has only one. However, in the case of  $\eta = 0.9$ , two distributions are almost the same. This is understandable since for  $\eta = 0.9$ ,  $L$  must be very big and the HGS are very close to the BS.

## 5 Conclusion

We have shown various properties of the *hypergeometric states*. The relationship with the BS is clarified together with the coherent state and the number state limits. The ladder operator formulation gives an algebraic characterization of the HGS based on the generally deformed oscillator algebra. The salient statistical properties of the HGS such as the sub-Poissonian character, the anti-bunching effect and the squeezing effects are investigated for a wide range of the parameters. The nonclassical features of the HGS for certain parameter ranges are demonstrated in terms of the quasiprobability

distributions, the Q-function and the Wigner function. On account of these remarkable properties we are tempted to think that the HGS play an important role in quantum optics. Surely they deserve further investigation including the method of generation.

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## Appendix. Hypergeometric Distribution

Consider a pot containing  $L_1$  red and  $L_2$  black balls. A group of  $M$  balls is chosen randomly. Then the probability  $q_n$  that the chosen group contains exactly  $n$  red balls is given by the hypergeometric distribution

$$q_n = \binom{L_1}{n} \binom{L_2}{M-n} \binom{L}{M}^{-1}, \quad L = L_1 + L_2.$$

It is easy to see  $q_n = |H_n^M(\eta, L)|^2$  (2.5) for  $\eta = L_1/L$ . The name is explained by the fact that the generating function of the above distribution can be expressed in terms of the hypergeometric functions [18]. Obviously the above distribution tends to the binomial distribution with probability  $\eta$  in the large  $L$  limit.

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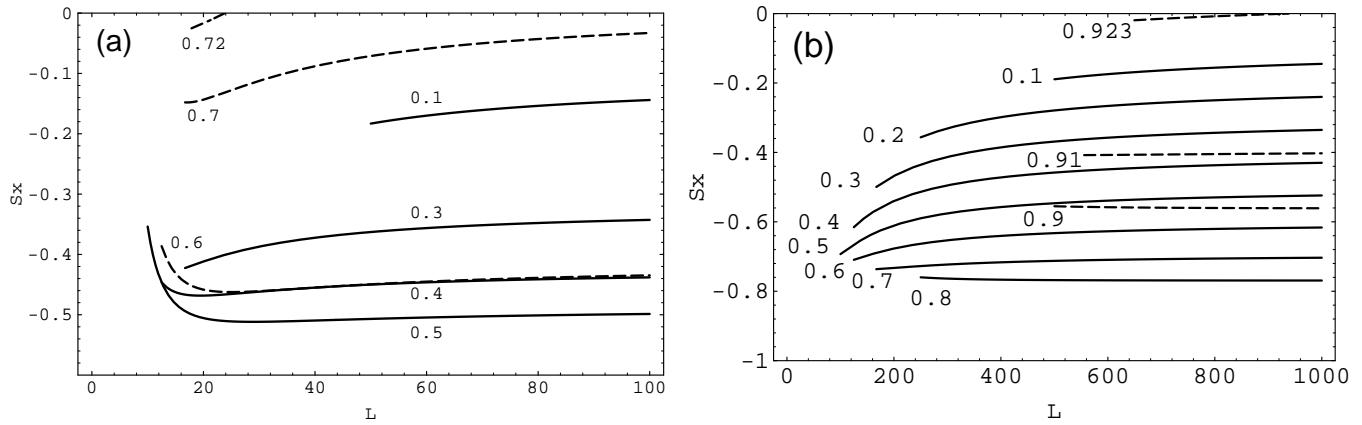


Figure 1: Squeezing index  $S_x$  of HGS as a function of  $L$ , for (a)  $M = 5$  and (b)  $M = 50$ . The  $\eta$  values are indicated in the figures.

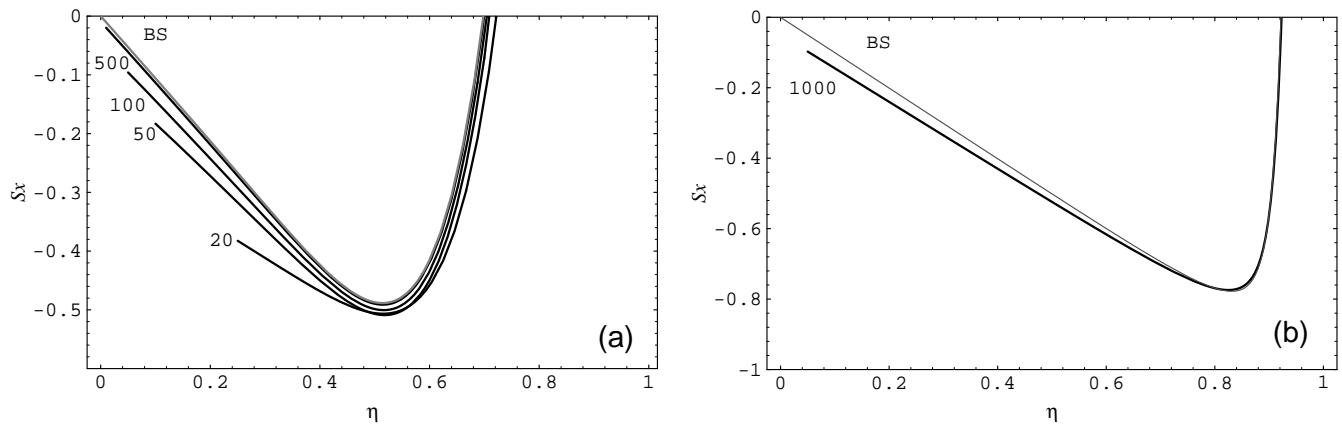


Figure 2: Squeezing index  $S_x$  of HGS as a function of  $\eta$  and for different  $M$  values: (a)  $M = 5$  and (b)  $M = 50$ . The  $L$  values are shown in the figures.

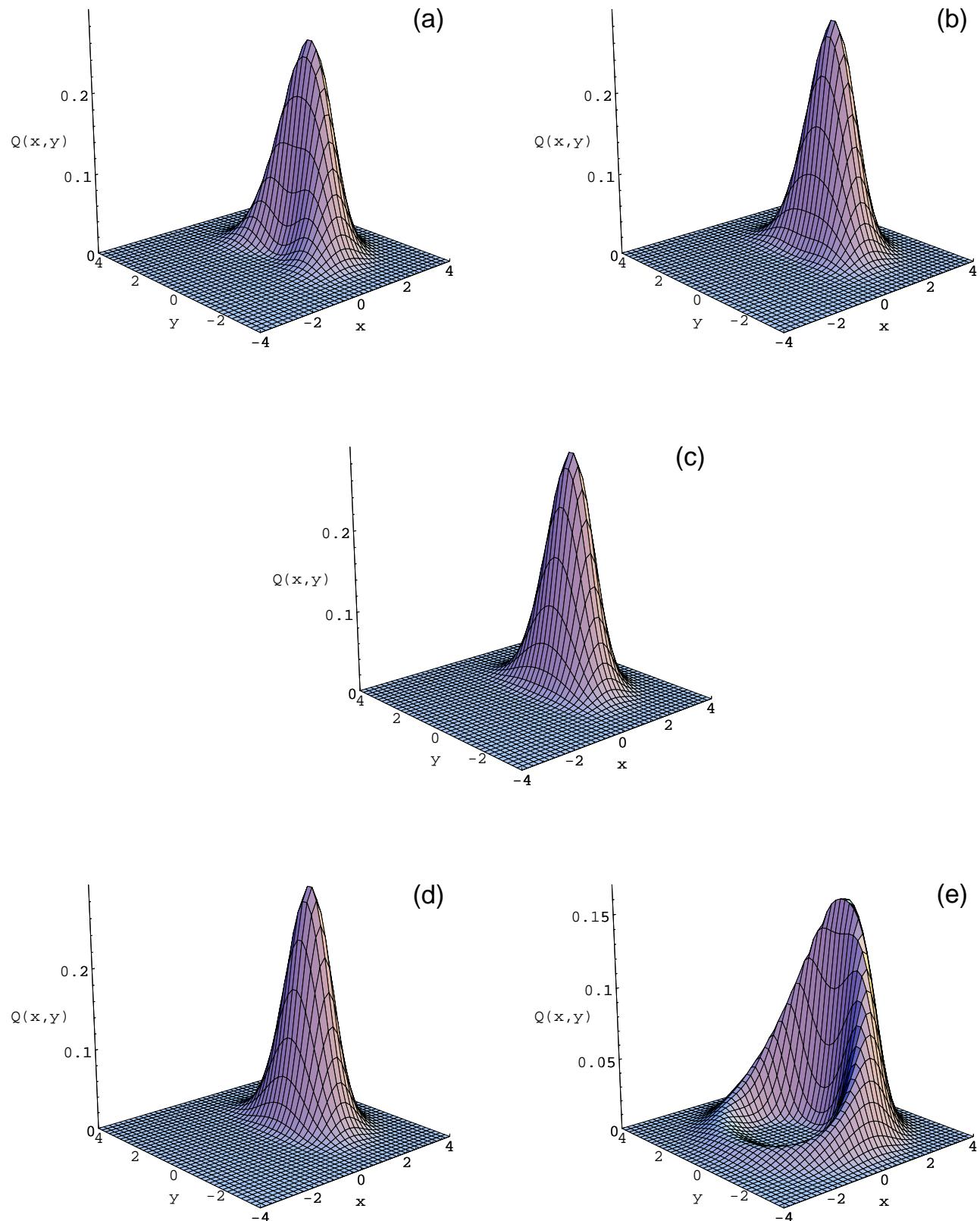


Figure 3:  $Q(\beta)$ -functions of HGS.  $\beta = x + iy$ . (a)  $\eta = 0.5, L = 10$ , (b)  $\eta = 0.5, L = 20$ , (c)  $\eta = 0.5, L = 40$ , (d)  $\eta = 0.5, L \rightarrow \infty$  (binomial state). Those four figures show how  $Q$ -function depends on  $L$ . Case (e) corresponds to  $\eta = 0.9$  and  $L = 50$ . In all the cases  $M = 5$ .

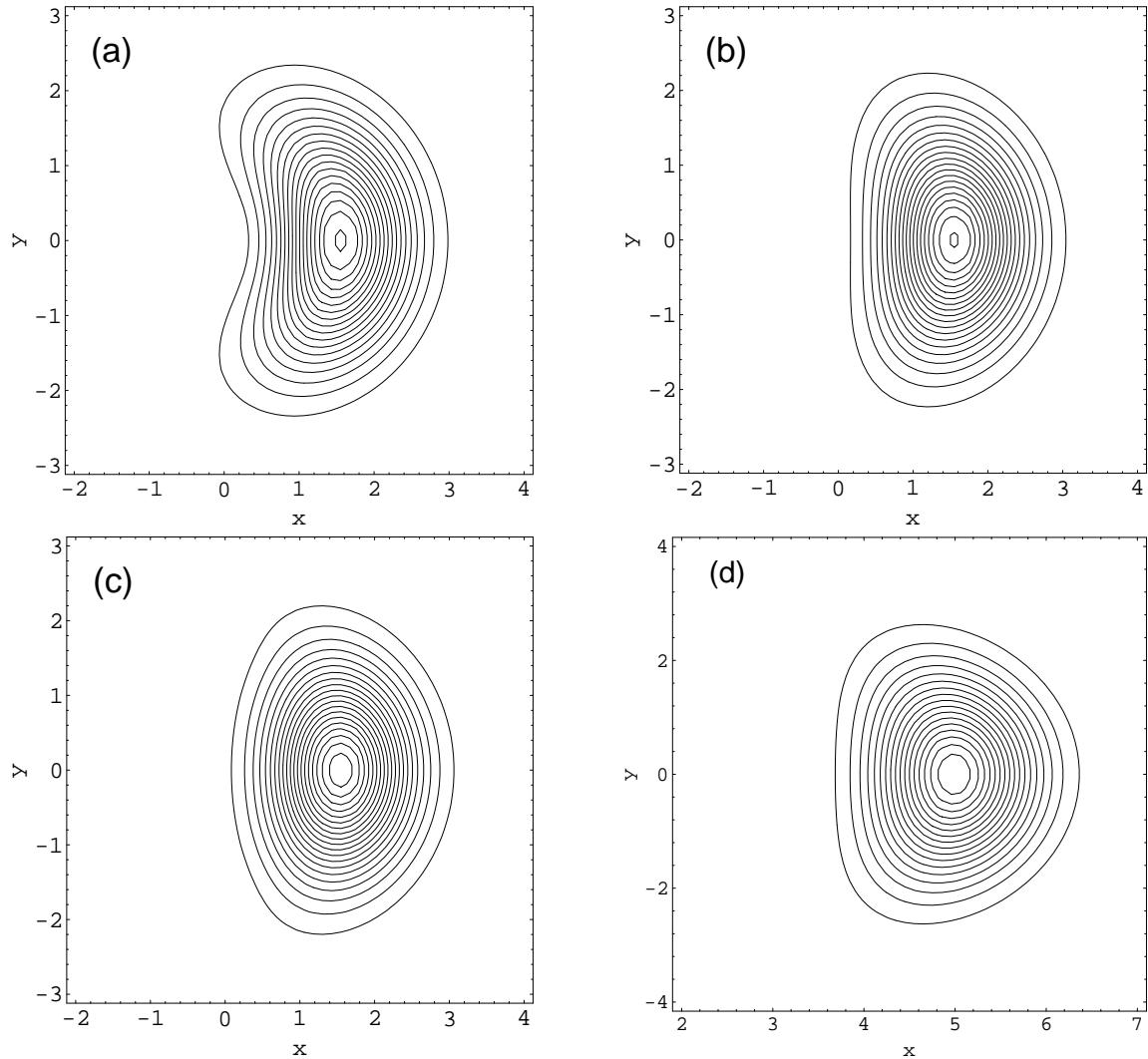


Figure 4: Contours of  $Q(\beta)$ -functions of HGS for  $M = 5$ ,  $\eta = 0.5$  and (a)  $L = 10$ , (b)  $L = 28$  and (c)  $L = \infty$  (Binomial state); (d)  $M = 50$ ,  $\eta = 0.5$  and  $L = 100$ .  $\beta = x + iy$ .

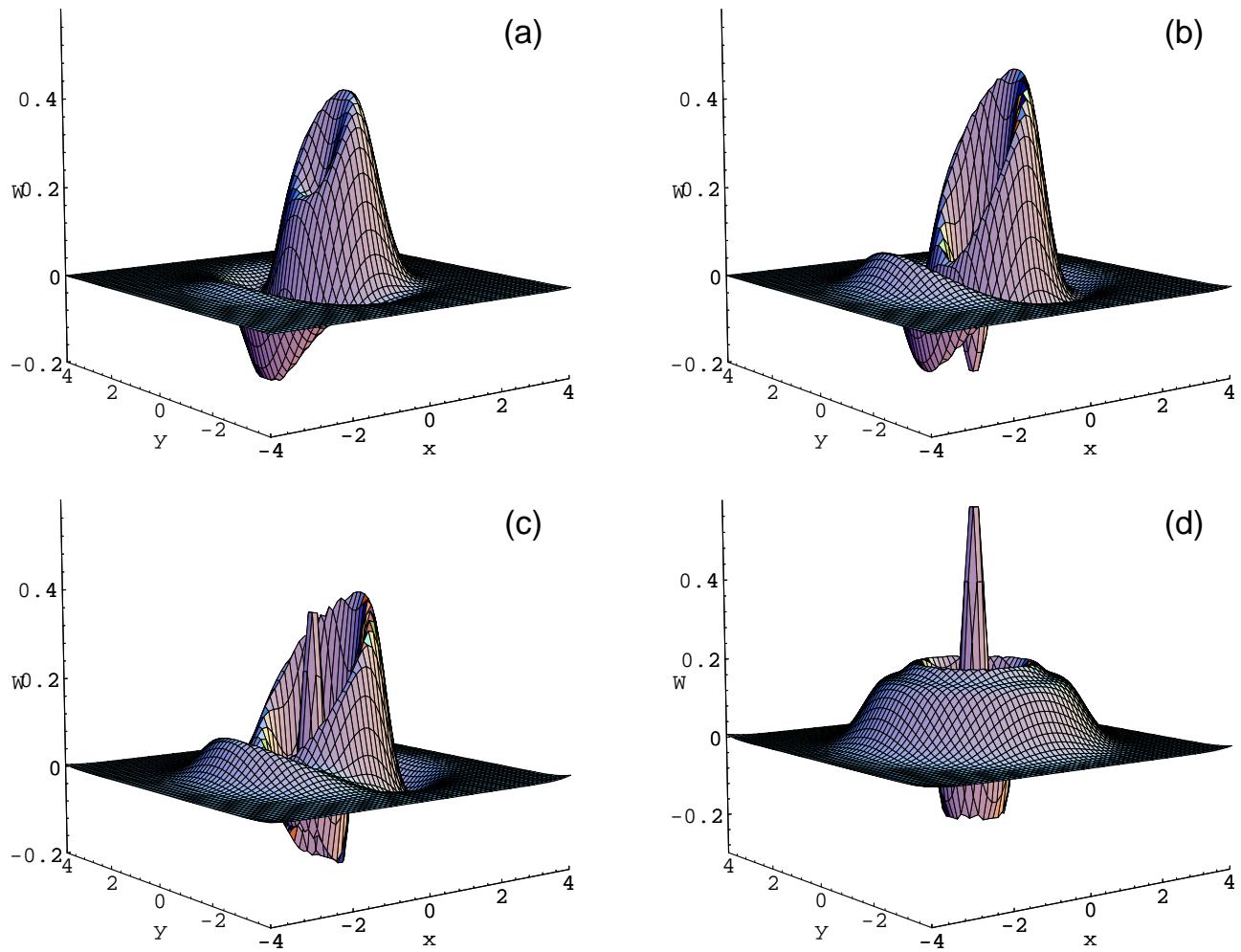


Figure 5:  $\beta = x + iy$ . Wigner function  $W(\beta)$  of HGS for  $M = 2$  and different  $\eta$  and  $L$ : (a)  $\eta = 0.2$ , (b)  $\eta = 0.5$ , (c)  $\eta = 0.9$  and (d)  $\eta = 1$  (the number state  $|2\rangle$ ). The smallest possible value of  $L$  is chosen for each  $\eta$  to show the maximal contrast with BS.

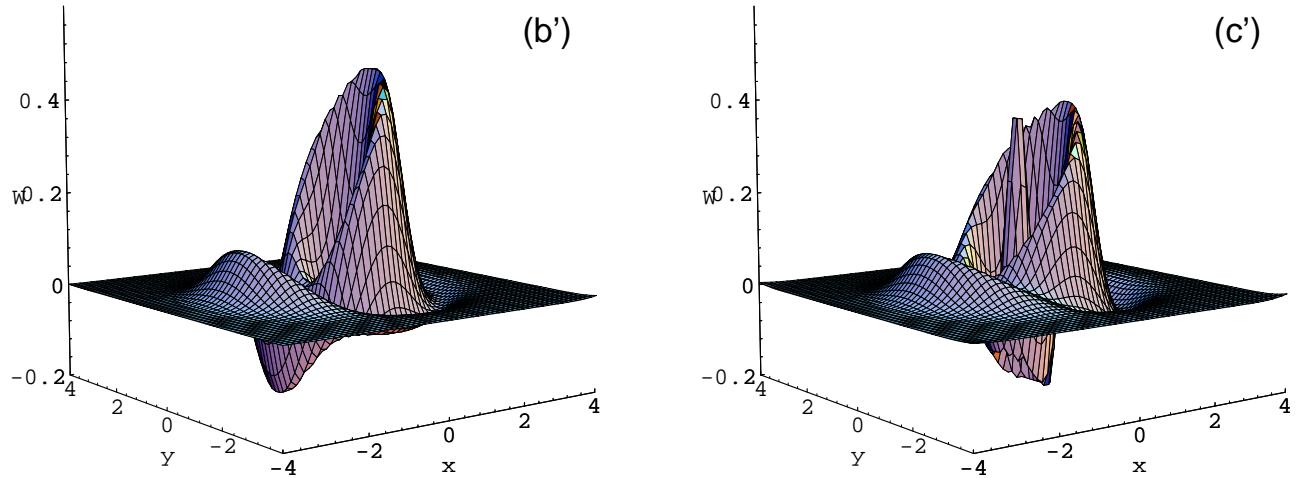


Figure 6: Wigner function of BS for  $M = 2$  and (b')  $\eta = 0.5$  and (c')  $\eta = 0.9$ .